Lecture note about $cov(\hat{\alpha},\hat{\beta}_2)$ to accompany Lecture 2 slide set

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This note is a translation of Appendix 3.A in BN. We include it as documentation and for completeness. If you are interested in this kind of exercise and can formulate a more elegant proof, let me know!

With reference to the notation in Lecture 2 we have

$$\begin{aligned} Cov\left(\hat{\alpha},\hat{\beta}_2\right) &= E\left[\left(\hat{\alpha}-\alpha\right)\left(\hat{\beta}_1-\beta_2\right)\right] = E\left[\hat{\alpha}\left(\hat{\beta}_2-\beta_2\right)-\alpha\left(\hat{\beta}_2-\beta_2\right)\right] \\ &= E\left[\hat{\alpha}\left(\hat{\beta}_2-\beta_2\right)\right] \qquad (1) \end{aligned}$$

and we want to show that $E\left[\hat{\alpha}\left(\hat{\beta}_2-\beta_2\right)\right]=0$.

Start but noting that $\hat{\beta}_2-\beta_2$:

$$\begin{aligned} \hat{\beta}_1-\beta_1 &= \hat{\beta}_1-E\left(\hat{\beta}_1\right) = \qquad (2) \\ &= \frac{\sum_{i=1}^{n} y_i\left(x_i-\bar{x}\right)}{n\hat{\sigma}_x^2} - \frac{\sum_{i=1}^{n} E\left(y_i\right)\left(x_i-\bar{x}\right)}{n\hat{\sigma}_x^2} \\ &= \frac{1}{n\hat{\sigma}_x^2}\sum_{i=1}^{n}\left[y_i-E\left(y_i\right)\right]\left(x_i-\bar{x}\right) \\ &= \frac{1}{n\hat{\sigma}_x^2}\sum_{i=1}^{n} e_i\left(x_i-\bar{x}\right), \end{aligned}$$

where we have used that

$$y_i-E\left(y_i\right)=y_i-\alpha-\beta_2\left(x_i-\bar{x}\right)=e_i.$$

Next, use the expression $\hat{\beta}_2-\beta_2$ in the definition of $cov\left(\hat{\alpha},\hat{\beta}_2\right)$ in (1):

$$\begin{aligned} E\left[\hat{\alpha}\left(\hat{\beta}_2-\beta_2\right)\right] &= E\left[\frac{1}{n}\sum_{j=1}^{n} y_j \frac{1}{n\hat{\sigma}_x^2}\sum_{i=1}^{n} e_i\left(x_i-\bar{x}\right)\right] \\ &= \frac{1}{n^2\hat{\sigma}_x^2} E\left[\sum_{j=1}^{n} y_j \sum_{i=1}^{n} e_i\left(x_i-\bar{x}\right)\right], \end{aligned}$$

where we have used $\hat{\alpha} = \bar{y}$.

Consider the case of $n = 2$: By inspection, the expression after the second equality sign becomes

$$\begin{aligned} &\frac{1}{4\hat{\sigma}_x^2} E\left[\sum_{j=1}^{2} y_j \sum_{i=1}^{2} e_i \left(x_i - \bar{x}\right)\right] \\ &= \frac{1}{4\hat{\sigma}_x^2} \left\{E\left[y_1 e_1 \left(x_1 - \bar{x}\right) + y_1 e_2 \left(x_2 - \bar{x}\right) + y_2 e_1 \left(x_1 - \bar{x}\right) + y_2 e_2 \left(x_2 - \bar{x}\right)\right]\right\}, \end{aligned}$$

i.e., the sum of all cross products between y_j and $e_i(x_i - \bar{x})$. A typical term in $\sum_{j=1}^{n} y_j \sum_{i=1}^{n} e_i \left(x_i - \bar{x}\right)$ is

$$\begin{aligned} E[y_j e_i \left(x_i - \bar{x}\right)] &= E\left\{\left[\alpha + \beta_2 \left(x_j - \bar{x}\right) + e_j\right] e_i \left(x_i - \bar{x}\right)\right\} \\ &= E\left[e_j e_i \left(x_i - \bar{x}\right)\right] \\ &= \begin{cases} 0 & \text{when } i \neq j \\ \sigma^2 \left(x_i - \bar{x}\right), & \text{when } i = j \ (n \text{ times}), \end{cases} \end{aligned}$$

By this argument, we see that the expression for $cov\left(\hat{\alpha}, \hat{\beta}_2\right)$ simplifies to

$$cov\left(\hat{\alpha}, \hat{\beta}_2\right) = \frac{\sigma^2}{n^2 \hat{\sigma}_x^2} \sum_{i=1}^{n} \left(x_i - \bar{x}\right) = 0. \tag{3}$$